Binomial Coefficients

Recall: The binomial coefficient for $0 \le k \le n$ is

$$\binom{n}{k} = \#\{S \subseteq [n] : \#S = k\}.$$

$\binom{n}{k}$ "n choose k"

Prop: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Saw before: Useful for counting, especially counting anagrams

Today: Relationships between binomial coefficients.

Useful Notation for Today: $C(k,n) := \{S \subseteq [n] : \#S = k\}$.

Prop: For all $0 \le k \le n$ we have $\binom{n}{k} = \binom{n}{n-k}$

pf: We construct a one-to-one correspondence between $C(k,n)$ and $C(n-k,n)$. Consider the assignment

$S \in C(k,n) \hookrightarrow [n] \setminus S$. We see for $S \subseteq [n]$ $\#S = k$ iff $\#([n] \setminus S) = n-k$. To see this is a one-to-one correspondence, note for $T \subseteq [n]$:

$x \in [n] \setminus ([n] \setminus T)$

iff $x \in [n]$ but $x \notin [n] \setminus T$

iff $x \in [n]$ but not $x \in [n] \setminus T$

iff $x \in [n] \wedge (\neg (x \in [n] \setminus T))$

iff $x \in [n] \wedge (\neg (x \in [n] \wedge x \notin T))$

iff $x \in [n] \wedge (x \notin [n] \vee x \in T)$

iff $(x \in [n] \wedge x \notin [n]) \vee (x \in [n] \wedge x \in T)$

iff $x \in [n] \wedge x \notin T$

iff $x \in T$

NB: Essentially we proved $A \setminus (A \setminus B) = A \cap B$.

NB: proved essentially $B \subseteq A$, then $A \setminus (A \setminus B) = B$

Hence, every $T \in C(n-k, n)$ can be obtained via our correspondence from $[n] \setminus T \in C(k, n)$. To see uniqueness of corresponding elements, $[n] \setminus T_1 = [n] \setminus T_2$ if and only if $T_1 = T_2$ for $T_1, T_2 \subseteq [n]$.

(Repeated Exercise: Prove If $A, B \subseteq C$, then $C \setminus A = C \setminus B$ iff $A = B$).

Hence, every element of $C(k,n)$ corresponds one-to-one with an element of $C(n-k, n)$, so by the Correspondence principle, we have:

$$\binom{n}{k} = \#C(k,n) = \#C(n-k,n) = \binom{n}{n-k}$$ ∎

Second Proof: Note that

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \binom{n}{n-k}$$ ∎

↳ More algebraic proof of the result ☺

Prop: For all $n \in \mathbb{N}$ we have $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

Pf: We count $\mathcal{P}([n])$ in two ways. For the first count, note we already showed $\#\mathcal{P}([n]) = 2^n$. For the second count, notice $\mathcal{P}[n] = \bigcup_{k=0}^{n} C(k,n)$ because each subset of $[n]$ has a uniquely determined cardinality. Moreover, $\{C(k,n) : 0 \le k \le n\}$ is a pairwise disjoint collection.

Hence applying the Sum Principle:

$$\#\mathcal{P}[n] = \#\bigcup_{k=0}^{n} C(k,n) = \sum_{k=0}^{n} \#C(k,n) = \sum_{k=0}^{n} \binom{n}{k}.$$

Finally we obtain $2^n = \#\mathcal{P}[n] = \sum_{k=0}^{n} \binom{n}{k}$ as desired. ☑

Very Hard Exercise: Prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ explicitly using $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Point: Combinatorial proof is a lot nicer here...

Prop: For all $0 \le k \le m \le n$ we have $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.

Pf: We count the following set in two ways:

$$\mathcal{F} = \{(A,B) : A \subseteq B \subseteq [n],\ \#A = k,\ \#B = m\}.$$

For our first count, we note every element of $\mathcal{F}$ can be constructed uniquely via the following two choices:

① Choose $B \subseteq [n]$ of size m (i.e. choose $B \in C(m,n)$) — $\binom{n}{m}$ options

② Choose $A \subseteq B$ of size k (correspondence $B \leftrightarrow [m]$ yields the $\binom{m}{k}$ options) — $\binom{m}{k}$ options

As number of options are independent of choices made, we see $\#\mathcal{F} = \binom{n}{m}\binom{m}{k}$ by the product principle.

For our second count, note every element of $\mathcal{F}$ can be constructed uniquely from the following choices:

① Choose $A \subseteq [n]$ of size k — $\binom{n}{k}$ options.

② Choose $D \subseteq [n] \setminus A$ of size $m-k$ — $\binom{n-k}{m-k}$ options.

The resulting element of $\mathcal{F}$ is $(A, A \cup D)$. By the Product principle, we obtain $\#\mathcal{F} = \binom{n}{k}\binom{n-k}{m-k}$. Putting these together we obtain $\binom{n}{m}\binom{m}{k} = \#\mathcal{F} = \binom{n}{k}\binom{n-k}{m-k}$ as desired. ☑

Prop (Pascal's Identity): For all $1 \le k \le n-1$ we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

NB: Pascal's Identity (together with $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$) yields a recursive definition of the binomial coefficients !!

Pf: Let $1 \le k \le n$ and note

$$C(k,n) = \{S \subseteq [n] : \#S = k,\ n \in S\} \cup \{S \subseteq [n] : \#S = k,\ n \notin S\}$$

Let $A = \{S \subseteq [n] : \#S = k,\ n \in S\}$ and

$B = \{S \subseteq [n] : \#S = k,\ n \notin S\}$.

Note $B = \{S \subseteq [n-1] : \#S = k\} = C(k, n-1)$.

On the other hand,

$$A = \{T \cup \{n\} \subseteq [n] : \#T = \underset{\downarrow}{k-1},\ T \subseteq [n-1]\},$$

which is in one-to-one correspondence with

$$\hat{A} = \{T \subseteq [n-1] : \#T = k-1\} = C(k-1, n-1).$$

Hence we obtain the following (completing the proof):

$$\binom{n}{k} = \#C(k,n) = \#(A \cup B) = (\#A) + (\#B)$$

$$= (\#\hat{A}) + (\#B) = (\#C(k-1,n-1)) + (\#C(k,n-1))$$

$$= \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Exercise: Try to find a formulaic proof of Pascal's Identity...

Prop (Binomial Theorem): Let $n \in \mathbb{N}_0$. In the expansion of $(x+y)^n$, the coefficient on $x^k y^{n-k}$ is $\binom{n}{k}$.

i.e. $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

NB: "$x+y$" is a binomial, so the binomial coefficients are so-named because they are coefficients of the binomial expansion $(x+y)^n$.

Pf: We proceed by induction on n.

Base Case: $(x+y)^0 = 1 = \binom{0}{0} = \sum_{k=0}^{0} \binom{0}{k} x^k y^{0-k}$

So when $n=0$, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Inductive Step: Suppose the result hold for some $n \geq 0$.

Now we compute as follows:

(Know: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$)

(WTS: $(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$)

$$(x+y)^{n+1} = (x+y)(x+y)^n$$

$$= (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= x \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}$$

$k = k' - 1$

$$= \sum_{k'-1=0}^{n} \binom{n}{k'-1} x^{(k'-1)+1} y^{n-(k'-1)} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}$$

$$= \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}$$

$$= \left(\sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \binom{n}{n+1-1} x^{n+1} y^{n+1-(n+1)} \right)$$

$$+ \left(\binom{n}{0} x^0 y^{n+1-0} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} \right)$$

$$= \binom{n}{0} x^0 y^{n+1-0} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) x^k y^{n+1-k} + \binom{n}{n} x^{n+1} y^{n+1-(n+1)}$$

$$= \binom{n+1}{0} x^0 y^{n+1-0} \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + \binom{n+1}{n+1} x^{n+1} y^{(n+1)-(n+1)}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{(n+1)-k}$$

NB: I went overtime finishing the proof... "

I'll make it up to you in the future,

I promise...